FACT SHEET: IMAGINARY AND COMPLEX NUMBERS

Definition: The number *i* is the imaginary square root of negative 1: $i = \sqrt{-1}$

In other words, $i^2 = -1$.

Simplifying a square root with a negative radicand:

Use the property $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ together with the above definition of i .

Ex: $\sqrt{-15} = \sqrt{-1 \cdot 15} = \sqrt{-1} \cdot \sqrt{15} = i\sqrt{15}$.

Ex: $\sqrt{-48} = \sqrt{-1 \cdot 16 \cdot 3} = \sqrt{-1} \cdot \sqrt{16} \cdot \sqrt{3} = i \cdot 4 \cdot \sqrt{3} = 4i\sqrt{3}$.

COMPLEX NUMBERS

Definitions: A ***complex number*** has the form ***a* + *bi***, where *a* and *b* are real numbers, and $i = \sqrt{-1}$. The "a" is called the ***real part***, and the "bi" is the ***imaginary part***.

Operations on Complex Numbers:

Addition and Subtraction: This is similar to adding and subtracting *like terms* with polynomials. You combine the real parts together, and the imaginary parts together. Think of the real and imaginary parts as "oil and water" – they do not mix together!

Ex's: Let $z_1 = 5 + 3i$ and $z_2 = 7 - 4i$.

Then $z_1 + z_2 = (5 + 3i) + (7 - 4i) = \underbrace{(5 + 7)}_{\textit{real parts}} + \underbrace{(3i - 4i)}_{\textit{imag. parts}} = 12 - i$

and $z_1 - z_2 = (5 + 3i) - (7 - 4i) = 5 + 3i - 7 - (-4i) = 5 + 3i - 7 + 4i$

Distribute the negative

$= \underbrace{(5 - 7)}_{\textit{real parts}} + \underbrace{(3i + 4i)}_{\textit{imag. parts}}$

$= -2 + 7i$.

Multiplication: This too is similar to working with polynomials. You may think of it as multiplying binomials, using the Distributive Law (or "FOIL"). As above, combine the real parts together as like terms, then the imaginary parts together as like terms.

The new challenge here is that whenever you have a term containing an "i^2", you must **use the fact that $\boldsymbol{i^2 = -1}$** . Therefore, that term becomes a *real term* (not imaginary).

Ex: $2i(10 + 9i) = (2i)(10) + (2i)(9i) = 20i + 18i^2 = 20i + 18(-1) = 20i - 18$

Distribute

$i^2 = -1$

Ex: Using $z_1 = 5 + 3i$ and $z_2 = 7 - 4i$ from above,

$$z_1 z_2 = (5 + 3i)(7 - 4i) = (5)(7) + (5)(-4i) + (3i)(7) + (3i)(-4i)$$

$$= 35 + -20i + 21i + -12i^2$$

combine $\quad i^2 = -1$

$$= 35 + i + (-12)(-1)$$

$$= 47 + i\ .$$

Rationalizing Denominators

Our "division" of complex numbers takes the form of fractions (remember $\frac{a}{b} = a \div b$).
It is bad form to leave an "i" in the denominator, so we "rationalize" the denominator.

Remember how to rationalize when there is a square root in the denominator? It works much in the same way, but you must also remember that $i^2 = -1$.

Recall: $\frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$ **Ex**: $\frac{5}{i} = \frac{5}{i} \cdot \frac{i}{i} = \frac{5i}{i^2} = \frac{5i}{-1} = -5i$.

To rationalize something of the form $\dfrac{a}{b+ci}$, multiply the top and bottom by the ***conjugate*** of the denominator (this means change the sign in front of the imaginary part).

$$\frac{a}{b+ci} = \frac{a}{(b+ci)} \cdot \frac{(b-ci)}{(b-ci)}$$ … and FOIL the denominators *carefully*.

Ex: $\frac{11}{5-4i} = \frac{11}{(5-4i)} \cdot \frac{(5+4i)}{(5+4i)} = \frac{11(5+4i)}{25 + 20i - 20i - 16i^2} = \frac{11(5+4i)}{25 - 16(-1)} = \frac{55+44i}{41}$ or $\frac{55}{41} + \frac{44}{41} i$.

cancel $\quad i^2 = -1$

Higher Powers of i

Remember the rules $a^m \cdot a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$? We will use these rules for higher powers of i .

First, $i^3 = i^2 \cdot i^1 = (-1)(i) = -i$. Then for i^4 , we have $i^4 = (i^2)^2 = (-1)^2 = 1$. Continuing in this way…

$i^1 = i$	$i^5 = i$	$i^9 = i$	$i^{13} = i$
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$	$i^{14} = -1$
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$	$i^{15} = -i$
$i^4 = 1$	$i^8 = 1$	$i^{12} = 1$	$i^{16} = 1$ …

Do you see the pattern? There are only 4 possible answers. We use the Laws of Exponents to figure out which of the 4 possible answers (or, try to figure out where it will appear on the above list).

Ex: $i^{22} = i^{20} \cdot i^2 = (i^4)^5 \cdot i^2 = (1)^5 \cdot -1 = -1$.

Ex: $i^{161} = i^{160} \cdot i^1 = (i^4)^{40} \cdot i = (1)^{40} \cdot i = i$.

Do you see why it is convenient to use i^4 whenever possible? It is convenient because $i^4 = 1$.