MATLAB Project: Using Eigenvalues to Study Spotted Owls Name________________________________

Purpose: To use eigenvalues and eigenvectors to understand the dynamics of this population and determine experimentally the critical rate for survival of juveniles to subadults—the value which that rate must equal or exceed for the population to survive. An extra credit problem asks for a deeper theoretical investigation and an exact calculation of that critical value.

Prerequisites: Sections 5.5 and 5.6

MATLAB functions used: * , \ , :, **sum**, **abs**, **for**, **eig**, **plot**; and **owldat** from Laydata Toolbox

ATTACH YOUR PLOTS AND TURN IN WITH THIS PAPER

Background. The spotted owls have three distinct life stages: juvenile (first year), subadult (second year) and adult (third year and older). Let $\mathbf{x_k} = \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 0 & .33 \\ t & 0 & 0 \\ 0 & .71 & .94 \end{bmatrix}$ where j_k, s_k and a_k denote the number of owls in each stage in year k and $\mathbf{x_{k+1}} = A\mathbf{x_k}$. As you may have seen in the earlier computer exercise on this topic, the owl population seems to die out eventually if t = .18 and seems to increase eventually if t = .30.

Some complex numbers occur in the eigenvalues and eigenvectors of *A*; to understand these better, read Section 5.5.

Definition. A *dominant eigenvalue* of a matrix *A* is an eigenvalue λ_1 of *A* such that $|\lambda_1| \geq |\lambda_i|$ for all eigenvalues λ_i of *A*.

It is true that the special type of matrix we are discussing here has only one dominant eigenvalue, so we will speak of "the" dominant eigenvalue λ_1. In fact, for all the matrices here, λ_1 is actually real and positive, so $|\lambda_1| = \lambda_1$.

1. Here you will experiment with several values of t to see how the dominant eigenvalue changes as t increases, and will find the "critical value."

To begin, type **owldat** to get the matrix for t = .18, $A = \begin{bmatrix} 0 & 0 & .33 \\ .18 & 0 & 0 \\ 0 & .71 & .94 \end{bmatrix}$. (You will also get $\mathbf{x0} = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$, for later use.) Then type the following lines:

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eig(A)              (a vector containing the eigenvalues of A)
abs(eig(A))         (a vector containing the modulus of each entry of eig(A) )
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You will see that the largest magnitude entry is 0.9836, so that is λ_1. Record it in the table below, under t = .18.

Next type **A(2,1) = .19** and use the up arrow key to execute the two lines above again. Repeat this for each value of t shown in the table below, and record the dominant eigenvalue each time:

Survival rate juv→subadult	t=	.18	.20	.22	.24	.25	.26	.28	.30
Dominant eigenvalue of A:	λ								

(b) Of the values you used for t, which is the smallest one for which $\lambda_1 \geq 1$? ________________ We will call this the "critical value" of t.

2. Now assign **A(2,1)** the value that you just found which causes t to have the critical value. Thus, the dominant eigenvalue λ_1 of your matrix *A* will be slightly larger than 1.

(a) Type **[V D] = eig(A)** , and record the columns of *V* below as $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$. Before doing that, notice the dominant eigenvalue of *A,* which we want to call λ_1, is the third diagonal entry of *D*, hence the third column of *V* is an eigenvector corresponding to λ_1. So record the third column of *V* as $\mathbf{v_1}$, and record the first two columns of *V* as $\mathbf{v_2}$ and $\mathbf{v_3}$:

$\mathbf{v_1} =$ $\mathbf{v_2} =$ $\mathbf{v_3} =$

Also record the eigenvalues of *A* that correspond to each of these columns:

$\lambda_1 =$ ______________ $\lambda_2 =$ ______________ $\lambda_3 =$ ______________

(b) (hand) It is true that $\{ \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \}$ is a basis for the vector space $\mathbf{C}^3$, so any vector can be written as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.[1] Let $\mathbf{x_0}$ denote an initial vector and define $\mathbf{x_k}=\mathbf{A}\mathbf{x_{k-1}}$. Suppose c_1, c_2 and c_3 are scalars such that $\mathbf{x_0} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$. Using this equation, explain what $\mathbf{x_k}$ will look like after k years, and why, if $c_1 \neq 0$, the population of owls will not die out. You must use the fact that $\lambda_1 \geq 1$.

(c) Let the initial vector be $\mathbf{x_0} = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$. Type the following lines to rearrange the columns of V so the eigenvector corresponding to λ_1 is the first column, and then to solve $\mathbf{x_0} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$ for the c_i 's:

V = V(:, [3 1 2]) (Create $V = [\mathrm{v_1}\ \mathrm{v_2}\ \mathrm{v_3}]$)
c = V\x0 (Solve $V_c = \mathbf{x_0}$ for **c**)

Record the coefficients: $c_1 =$ ______________ $c_2 =$ ______________ $c_3 =$ ______________

Notice c_1 is not zero, so the owl population will definitely not die out when $\mathbf{x_0} = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$ is the initial vector.

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[1] The space $\mathbf{C}^3$ is much like $\mathbf{R}^3$: its elements are all triples of complex numbers, and its scalars are the complex numbers. Also, {(1,0,0), (0,1,0), (0,0,1)} is a basis for $\mathbf{C}^3$ so its dimension is three, hence any three independent vectors in $\mathbf{C}^3$ form a basis for the space. Finally, the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ found in question 2 are independent -- you can check that directly, or just notice that the three eigenvalues of A are distinct.

3. Continue to use $\mathbf{x_0} = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$ as the initial vector. Choose two values of t: t_1 less than the critical value you found above, and t_2 greater than that critical value. Record the values you choose:

t_1 = ________ t_2 = ________

(a) Using $t = t_1$, calculate and plot the values of j_k, s_k and a_k from 1997 until 2020. The following commands will do these things for $t = t_1$:

A(2,1) = (your value for t_1)
x = x0; P = x; for i = 1997:2020, x = A*x; P = [P x]; end
yr = 1996:2020; plot(yr, P)

Print this graph. Type **help print** if you need instructions. Label the graph with your value of t_1 and label the individual curves "adult," "subadult," and "juvenile."

(b) Repeat the calculations, printing and labeling using your value of t_2.

(c) Discuss: what long term population trends do your graphs show in the three age groups when $t = t_1$, and what trends when $t = t_2$? Are these the results you expected, based on what you know about the dominant eigenvalue of the matrix A in each case?

4. (Extra credit, all hand work. Use your paper and attach.) Let $A = \begin{bmatrix} 0 & 0 & a \\ t & 0 & 0 \\ 0 & b & c \end{bmatrix}$, and assume a,b,c,t are positive.

(a) Let $f(\lambda)$ denote the characteristic polynomial of A. Calculate it and show work. You should get $f(\lambda) = -\lambda^3 + c\,\lambda^2 + abt$.

(b) Prove that A has only one real eigenvalue, that it is positive, and that the other two eigenvalues of A must be conjugate complex numbers. Let λ_1 denote the real positive eigenvalue and let λ_2 and λ_3 denote the other two eigenvalues.

Hint: Since $y = f(\lambda)$ has only real coefficients, you can sketch its graph in R^2. It will be helpful to calculate its y-intercept and to use the derivative to find the turning points. Use this graph to explain why there is only one real zero of $f(\lambda)$ and it is positive. Then use things you know about zeros of polynomials to explain why the other two zeros must be conjugate complex numbers.

(c) Prove that λ_1 is greater than $|\lambda_2| = |\lambda_3|$, hence the real positive eigenvalue of A will always be the dominant eigenvalue for this type matrix.

Hint: Explain first why $f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$ is true and use that to explain why $\lambda_1\lambda_2\lambda_3$ equals abt; explain next why $\lambda_2\lambda_3$ equals $|\lambda_2|^2$; thus $\lambda_1|\lambda_2|^2 = abt$; finally, explain why $\lambda_1{}^3 = c\lambda_1{}^2 + abt$ is true. Then put this information together.

(d) Assume $\lambda_1 = 1$ and use this to obtain a formula for the exact critical value of t. Evaluate your formula when $a = .33$, $b = .71$ and $c = .94$, and compare this with the critical value you found experimentally in question 1. Are they essentially the same?

Discuss what $\lambda_1 = 1$ means in the owl example. Does it mean no births or deaths? If not, what does it mean?